Automorphism groups of Witt algebras *

Jianzhi Han, Yucai Su †
Department of Mathematics, Tongji University, Shanghai 200092, China

Abstract: The automorphism groups $\mathrm{Aut}\,A_n$ and $\mathrm{Aut}\,W_n$ of the polynomial algebra $A_n = \mathbb{C}[x_1, x_2, \cdots, x_n]$ and the rank n Witt algebra $W_n = \mathrm{Der}\,A_n$ are studied in this paper. It is well-known that $\mathrm{Aut}\,A_n$ for $n \geq 3$ and $\mathrm{Aut}\,W_n$ for $n \geq 2$ are open. In the present paper, by characterizing the semigroup $\mathrm{End}\,W_n\backslash\{0\}$ of nonzero endomorphisms of W_n via the semigroup of the so-called Jacobi tuples, we establish an isomorphism between $\mathrm{Aut}\,A_n$ and $\mathrm{Aut}\,W_n$ for any positive integer n. In particular, this enables us to work out the automorphism group $\mathrm{Aut}\,W_2$ of W_2.

Key words: polynomial algebra, Witt algebra, automorphism, Jacobi conjecture, Jacobi tuple.

Mathematics Subject Classification (2010): 17B40, 16S50.

1 Introduction

With a history of 100 years [1], the (one-sided rank n) Witt algebras $W_n = \mathrm{Der}\,A_n$ (the derivation algebras of the polynomial algebras $A_n := \mathbb{C}[x_1, ..., x_n]$ of n variables for all $n \geq 1$) are the first known examples of infinite-dimensional simple Lie algebras. However, the determination of automorphism groups $\mathrm{Aut}\,W_n$ of W_n is a long outstanding open problem (even for case $n = 2$). It is well-known (e.g., [4, 11–17]) that automorphism groups of Lie algebras constitute an important part in the structure theory of Lie algebras. For the case of the two-sided Witt algebra $W_n^{\pm} = \mathrm{Der}\,A_n^{\pm}$ (the derivation algebra of the Laurent polynomial algebra $A_n^{\pm} := \mathbb{C}[x_1^{\pm1}, ..., x_n^{\pm1}]$), the problem of determining the automorphism group $\mathrm{Aut}\,W_n^{\pm}$ of $W_n^{\pm}$ is much easier (e.g., [2, 12–14, 22]), as any automorphism $\sigma \in \mathrm{Aut}\,W_n^{\pm}$ must fix the set $\mathscr{F}_{W_n^{\pm}}$ of the ad-locally finite elements of $W_n^{\pm}$ and in this case $\mathscr{F}_{W_n^{\pm}}$ turns out to be the vector space $\mathscr{F}_{W_n^{\pm}} = \oplus_{i=1}^{n}\mathbb{C}x_i\frac{\partial}{\partial x_i}$. In sharp contrast to $W_n^{\pm}$, the set $\mathscr{F}_{W_n}$ of the ad-locally finite elements of W_n is unachievable.

The distinguished Jacobi conjecture posed by Keller in 1939 says that if $f_1, f_2, ..., f_n \in A_n$ are n polynomials on n variables such that the corresponding Jacobi determinant $J(f_1, ..., f_n) := \mathrm{Det}\big(\frac{\partial f_i}{\partial x_j}\big)_{1\leq i,j\leq n} \in \mathbb{C}^{\times} := \mathbb{C}\backslash\{0\}$ is a nonzero complex number (in this case, the n-tuple $(f_1, ..., f_n)$ is referred to as a Jacobi tuple in the present paper), then $f_1, f_2, ..., f_n$ are generators of A_n, namely, $A_n = \mathbb{C}[f_1, f_2, ..., f_n]$. Many interesting results would follow if this conjecture holds. Unfortunately, over seven decades' endeavor made by many mathematicians (e.g., [3, 5, 7, 10, 18–21]), it is still an open problem. Obviously, the Jacobi conjecture is equivalent to the statement that every endomorphism of A_n sending the generating tuple

* Supported by NSF grant no. 11371278, 11431010, the Fundamental Research Funds for the Central Universities of China, Innovation Program of Shanghai Municipal Education Commission and Program for Young Excellent Talents in Tongji University.

† Correspondence: Y. Su (Email: ycsu@tongji.edu.cn)

$(x_1, ..., x_n)$ to a Jacobi tuple is an automorphism. Thus the Jacobi conjecture is closely related to the automorphism group $\mathrm{Aut}\, A_n$ of A_n. The group $\mathrm{Aut}\, A_n$ is clear in case $n \le 2$ (cf. [6]), but for $n \ge 3$ this is yet undetermined. Obviously, there are three types automorphisms: s_i, τ_a, ψ_p for $1 \le i \le n-1$, $a \in \mathbb{C}^\times$, $p \in \mathbb{Z}^{\ge 0}$, where s_i is the automorphism which switches x_i and x_{i+1} and fixes other x_j's, while τ_a is the automorphism which sends x_1 to ax_1 and fixes other x_j's, and ψ_p is the automorphism which sends x_2 to $x_2 + x_1^p$ and fixes other x_j's. The subgroup $\mathrm{Ta}\, A_n$ of $\mathrm{Aut}\, A_n$ generated by these three types automorphisms is the group of tame automorphisms, and the elements of $\mathrm{Aut}\, A_n \backslash \mathrm{Ta}\, A_n$ are called wild automorphisms. It is well-known that there are no wild automorphisms of A_2. The first example of a wild automorphism is the Nagata automorphism σ_1 of A_3 given in [6] (and proved to be wild in [8, 9]) as follows:

$$\sigma_1(x_1) = x_1 - 2(x_2^2 + x_1x_3)x_2 - (x_2^2 + x_1x_3)^2 x_3, \quad \sigma_1(x_2) = x_2 + (x_2^2 + x_1x_3)x_3, \quad \sigma_1(x_3) = x_3.$$

The Jacobi conjecture is also closely related to another conjecture posed in [22, Conjecture 1] (referred to as the Witt algebra's conjecture for easy reference) which states that any nonzero endomorphism of W_n is an automorphism (or equivalently, any nonzero endomorphism of W_n is surjective), namely, $\mathrm{Aut}\, W_n = \mathrm{End}\, W_n \backslash \{0\}$. In fact, it was proved in [22, Theorem 4.1] that the Witt algebra's conjecture implies the Jacobi conjecture. From this, one can expect that the determination of $\mathrm{Aut}\, W_n$ is a highly nontrivial problem.

In the present paper, by embedding the Witt algebra W_n for any $n \ge 1$ into the derivation algebra $\bar{W}_n = \mathrm{Der}\, \bar{A}_n$ of the field $\bar{A}_n = \mathbb{C}(x_1, ..., x_n)$ of rational functions in n variables (regarding $\bar{A}_n$ as an algebra over $\mathbb{C}$), we characterize the semigroup $\mathrm{End}\, W_n \backslash \{0\}$ via the set JT_n of Jacobi tuples of A_n. This provides us a way to prove an equivalence between the Jacobi conjecture and the Witt algebra's conjecture, and to establish an isomorphism between $\mathrm{Aut}\, W_n$ and $\mathrm{Aut}\, A_n$. The later result in turn enables us to work out the automorphism group $\mathrm{Aut}\, W_2$ of W_2.

To summarize our main results, we first give a semigroup structure on JT_n by defining for $f = (f_1, ..., f_n)$, $g = (g_1, ..., g_n) \in \mathrm{JT}_n$,

$$f \cdot g = h, \text{ where } h = (h_1, ..., h_n) \text{ with } h_i = g_i(f_1, ..., f_n) \text{ for } 1 \le i \le n, \tag{1.1}$$

where $g_i(f_1, ..., f_n) = g_i|_{(x_1, \cdots, x_n) = (f_1, \cdots, f_n)}$. By the chain rule of partial derivatives, we see that the resulting tuple h is indeed in JT_n, and obtain a semigroup JT_n under the multiplication "$\cdot$" defined in (1.1).

Let $f = (f_1, f_2, \cdots, f_n) \in \mathrm{JT}_n$ be a Jacobi tuple, and assume $J(f_1, f_2, \cdots, f_n) = c \in \mathbb{C}^\times$. Let σ_f be the linear map of W_n by defining for $k_i \in \mathbb{Z}^{\ge 0}$ and $1 \le j \le n$,

$$\sigma_f(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \partial_j) = f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n} \theta_j, \text{ where } \partial_j = \frac{\partial}{\partial x_j}, \ \theta_j = \frac{1}{c} \sum_{l=1}^{n} M_{lj} \partial_l, \tag{1.2}$$

and M_{lj} is the (l,j)-cofactor of the Jacobi matrix $M := \left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le n}$. One can easily verify that $\theta_j(f_i) = \delta_{i,j}$ for $1 \le i,j \le n$, from this it is easy to check that σ_f is a nonzero endomorphism of the Lie algebra W_n. Thus we obtain a semigroup homomorphism

$$\xi : \mathrm{JT}_n \to \mathrm{End}\, W_n \backslash \{0\} \text{ sending } f \mapsto \sigma_f. \tag{1.3}$$

Let $\tau \in \mathrm{Aut}\, A_n$. Then we have a Jacobi tuple $f_\tau := (\tau(x_1), ..., \tau(x_n)) \in \mathrm{JT}_n$, thus τ corresponds to a nonzero endomorphism $\sigma_{f_\tau} \in \mathrm{End}\, W_n \backslash \{0\}$, and we obtain a semigroup homomorphism

$$\zeta : \mathrm{Aut}\, A_n \to \mathrm{End}\, W_n \backslash \{0\} \text{ sending } \tau \mapsto \sigma_{f_\tau}. \tag{1.4}$$

Now we can summarize our main results as follows.

Theorem 1.1. (1) *The Jacobi conjecture is equivalent to the Witt algebra's conjecture.*

(2) *The map in* (1.3) *is a semigroup isomorphism* $\xi : \mathrm{JT}_n \cong \mathrm{End}\, W_n \backslash \{0\}$.

(3) *The map in* (1.4) *induces a group isomorphism* $\zeta : \mathrm{Aut}\, A_n \cong \mathrm{Aut}\, W_n$.

(4) *The group* $\mathrm{Aut}\, W_2$ *is generated by* s, τ_a, ψ_p *for* $a \in \mathbb{C}^\times$, $p \in \mathbb{Z}^{\ge 0}$, *where,*

$$\begin{aligned}
&s(x_1^i x_2^j \partial_1 + x_1^k x_2^l \partial_2) = x_2^i x_1^j \partial_2 + x_2^k x_1^l \partial_1, \\
&\tau_a(x_1^i x_2^j \partial_1 + x_1^k x_2^l \partial_2) = a^i x_1^i x_2^j \partial_1 + a^k x_1^k x_2^l \partial_2, \\
&\psi_p(x_1^i x_2^j \partial_1 + x_1^k x_2^l \partial_2) = x_1^i (x_2 + x_1^p)^j (\partial_1 - x_1^{p-1} \partial_2) + x_1^k (x_2 + x_1^p)^l \partial_2,
\end{aligned}$$

for $i, j, k, l \in \mathbb{Z}^{\ge 0}$.

Finally we remark that the isomorphism in Theorem 1.1 (3) may provide a possible way to study automorphisms of A_n using the theory of Lie algebras. This is also our goal in a sequel.

2 Some lemmas

Let n be a positive integer (we assume $n \ge 2$). Denote by $\underline{n}$, $\mathbb{Z}^{\ge 0}$ and $\mathbb{C}^\times$ the set $\{1, 2, \cdots, n\}$, the set of all non-negative integers and the set of non-zero complex numbers, respectively. Let $A_n = \mathbb{C}[x_1, x_2, \cdots, x_n]$ be the polynomial algebra of n variables $x_1, x_2, \cdots, x_n$ over complex field $\mathbb{C}$. Denote $W_n = \mathrm{Der}\, A_n$, the derivation algebra of A_n.

It is well known that W_n is the free A_n-module of rank n with basis $\{\partial_i \mid i \in \underline{n}\}$:

$$W_n = \bigoplus_{i \in \underline{n}} A_n \partial_i = \left\{ \sum_{i=1}^n P_i \partial_i \;\middle|\; P_i \in A_n \right\}, \quad \text{where } \partial_i = \frac{\partial}{\partial x_i}.$$

Let $\bar{A}_n = \mathbb{C}(x_1, x_2, \cdots, x_n)$ be the quotient field of A_n, and $\bar{W}_n = \operatorname{Der} \bar{A}_n$ the corresponding derivation algebra of $\bar{A}_n$ (regarding $\bar{A}_n$ as a $\mathbb{C}$-algebra). Then obviously, $\bar{W}_n$ is the n-dimensional $\bar{A}_n$-vector space with basis $\{\partial_i \,|\, i \in \underline{n}\}$:

$$\bar{W}_n = \bigoplus_{i \in \underline{n}} \bar{A}_n \partial_i = \left\{ \sum_{i=1}^{n} P_i \partial_i \,\middle|\, P_i \in \bar{A}_n \right\}, \text{ and } W_n \subset \bar{W}_n.$$

Note that the space $\bar{W}_n \oplus \bar{A}_n$ is a Lie subalgebra of the Weyl type Lie algebra $\mathscr{W}_n$, where $\mathscr{W}_n$ is the Lie algebra consisting of all differential operators on $\bar{A}_n$. In particular, for any $a_1, a_2 \in \bar{A}_n, D_1, D_2 \in \bar{W}_n$, one has

$$[a_1D_1, a_2D_2] = [a_1D_1, a_2]D_2 + a_2[a_1D_1, D_2] = a_1D_1(a_2)D_2 - a_2D_2(a_1)D_1 + a_1a_2[D_1, D_2]. \quad (2.1)$$

Let $\sigma \in \operatorname{End} W_n \backslash \{0\}$ (the set of nonzero endomorphisms of the Lie algebra W_n). Then $\operatorname{Ker} \sigma = 0$ as the ideal generated by a single nonzero element in $\operatorname{Ker} \sigma$ would be W_n itself. Denote

$$\theta_i = \sigma(\partial_i) \in W_n \ \text{ for } \ i \in \underline{n}. \quad (2.2)$$

The following is the technical lemma in obtaining our main results.

Lemma 2.1. *The elements $\theta_1, ..., \theta_n$ are $\bar{A}_n$-linear independent.*

Proof. Suppose conversely that there exists $I_0 \subsetneq \underline{n}$ such that $\{\theta_i \mid i \in I_0\}$ forms a maximal $\bar{A}_n$-linearly independent subset of $\{\theta_i \mid i \in \underline{n}\}$. Choose any $i_1 \in \underline{n} \backslash I_0$, and assume that

$$\theta_{i_1} = \sum_{j \in I_0} b_j \theta_j \ \text{ for some } b_j \in \bar{A}_n. \quad (2.3)$$

Let $k \in \mathbb{Z}^{\geq 0}$ and denote $D_k = \frac{1}{k+2}\sigma(x_{i_1}^{k+2}\partial_{i_1})$. We have

$$[\theta_j, D_k] = \sigma\Big(\Big[\partial_j, \frac{1}{k+2}x_{i_1}^{k+2}\partial_{i_1}\Big]\Big) = 0 \ \text{ for any } \ j \in I_0. \quad (2.4)$$

Applying σ to $x_{i_1}^{k+1}\partial_{i_1} = [\partial_{i_1}, \frac{1}{k+2}x_{i_1}^{k+2}\partial_{i_1}]$, by (2.1)–(2.4), we obtain

$$\sigma(x_{i_1}^{k+1}\partial_{i_1}) = [\theta_{i_1}, D_k] = \sum_{j \in I_0} [b_j\theta_j, D_k] = \sum_{j \in I_0} c_{jk}\theta_j, \ \text{ where } c_{jk} = -D_k(b_j) \in \bar{A}_n. \quad (2.5)$$

Using this and the fact that $[\theta_i, \theta_j] = 0$ for $i, j \in \underline{n}$, we have

$$0 = \sigma([\partial_l, x_{i_1}^{k+1}\partial_{i_1}]) = \sum_{j \in I_0} [\theta_l, c_{jk}\theta_j] = \sum_{j \in I_0} \theta_l(c_{jk})\theta_j \ \text{ for any } l \in I_0. \quad (2.6)$$

This together with the $\bar{A}_n$-linear independence of $\{\theta_i \mid i \in I_0\}$ implies that $\theta_l(c_{jk}) = 0$ for all $j, l \in I_0$ and $k \in \mathbb{Z}^{\geq 0}$. From this and (2.1), we obtain

$$(k_2 - k_1)\sigma(x_{i_1}^{k_1+k_2+1}\partial_{i_1}) = [\sigma(x_{i_1}^{k_1+1}\partial_{i_1}), \sigma(x_{i_1}^{k_2+1}\partial_{i_1})] = 0 \ \text{ for any } k_1, k_2 \in \mathbb{Z}^{\geq 0},$$

contradicting the fact that $\operatorname{Ker} \sigma = 0$. $\square$

Lemma 2.2. *For any $i, j \in \underline{n}$ and $k = 1, 2$, there exists $a_i \in \bar{A}_n$ such that $\sigma(x_i^k \partial_j) = a_i^k \theta_j$.*

Proof. For any $k \in \mathbb{Z}^{\geq 0}$, assume that $\sigma(x_i^k \partial_j) = \sum_{l=1}^n a_{ijl}^{(k)} \theta_l$ for some $a_{ijl}^{(k)} \in \bar{A}_n$. For $i, j, m \in \underline{n}$ and $1 \leq k \in \mathbb{Z}^{\geq 0}$, we have

$$\begin{aligned}
-\delta_{i,m} \sum_{l=1}^n k a_{ijl}^{(k-1)} \theta_l &= -\delta_{i,m} k \sigma(x_i^{k-1} \partial_j) = \sigma([x_i^k \partial_j, \partial_m]) \\
&= [\sigma(x_i^k \partial_j), \sigma(\partial_m)] = \sum_{l=1}^n [a_{ijl}^{(k)} \theta_l, \theta_m] \\
&= -\sum_{l=1}^n \theta_m(a_{ijl}^{(k)}) \theta_l
\end{aligned}$$

by (2.1) and $[\theta_i, \theta_j] = 0$ for $i, j \in \underline{n}$. Hence by Lemma 2.1,

$$\theta_m(a_{ijl}^{(k)}) = \delta_{im} k a_{ijl}^{(k-1)} \quad \text{for all } i, j, l, m \in \underline{n} \text{ and } 1 \leq k \in \mathbb{Z}^{\geq 0}. \tag{2.7}$$

In particular,

$$\theta_m(a_{ijl}^{(1)}) = \delta_{im} \delta_{jl} \quad \text{for all } i, j, l, m \in \underline{n}, \tag{2.8}$$

since $a_{ijl}^{(0)} = \delta_{jl}$ (the Kronecker delta). For simplicity, denote $a_{ijl} = a_{ijl}^{(1)}$ for any $i, j, l \in \underline{n}$.

Claim 1. *We have $\sigma(x_i \partial_j) = a_i \theta_j$ and $\theta_j(a_i) = \delta_{ij}$ for $i, j \in \underline{n}$, where $a_i = a_{ill}$ for all $l \in \underline{n}$.*

Using (2.1) and (2.8), for arbitrary $i_1, i_2, j_1, j_2 \in \underline{n}$ we have

$$\begin{aligned}
&\sum_{l=1}^n (\delta_{j_1 i_2} a_{i_1 j_2 l} - \delta_{j_2 i_1} a_{i_2 j_1 l}) \theta_l \\
= \;& \delta_{j_1 i_2} \sigma(x_{i_1} \partial_{j_2}) - \delta_{j_2 i_1} \sigma(x_{i_2} \partial_{j_1}) = \sigma([x_{i_1} \partial_{j_1}, x_{i_2} \partial_{j_2}]) \\
= \;& [\sigma(x_{i_1} \partial_{j_1}), \sigma(x_{i_2} \partial_{j_2})] = \sum_{l_1, l_2 = 1}^n [a_{i_1 j_1 l_1} \theta_{l_1}, a_{i_2 j_2 l_2} \theta_{l_2}] \\
= \;& \sum_{l_1, l_2 = 1}^n \big(a_{i_1 j_1 l_1} \theta_{l_1}(a_{i_2 j_2 l_2}) \theta_{l_2} - a_{i_2 j_2 l_2} \theta_{l_2}(a_{i_1 j_1 l_1}) \theta_{l_1} \big) \\
= \;& \sum_{l_1, l_2 = 1}^n (\delta_{l_1 i_2} \delta_{j_2 l_2} a_{i_1 j_1 l_1} \theta_{l_2} - \delta_{l_2 i_1} \delta_{j_1 l_1} a_{i_2 j_2 l_2} \theta_{l_1}) \\
= \;& a_{i_1 j_1 i_2} \theta_{j_2} - a_{i_2 j_2 i_1} \theta_{j_1}.
\end{aligned}$$

By Lemma 2.1 again,

$$\delta_{j_1 i_2} a_{i_1 j_2 l} - \delta_{j_2 i_1} a_{i_2 j_1 l} = \delta_{j_2 l} a_{i_1 j_1 i_2} - \delta_{j_1 l} a_{i_2 j_2 i_1} \quad \text{for any } i_1, i_2, j_1, j_2, l \in \underline{n}. \tag{2.9}$$

Setting $j_1 = j_2 = j$ in (2.9), one has

$$\delta_{j i_2} a_{i_1 j l} - \delta_{j i_1} a_{i_2 j l} = \delta_{jl} (a_{i_1 j i_2} - a_{i_2 j i_1}) \quad \text{for any } i_1, i_2, j, l \in \underline{n},$$

which is equivalent to

$$\delta_{ji_2}a_{i_1jl} - \delta_{ji_1}a_{i_2jl} = 0 \quad \text{for any } i_1, i_2, j \neq l \in \underline{n} \tag{2.10}$$

and

$$\delta_{ji_2}a_{i_1jj} - \delta_{ji_1}a_{i_2jj} = a_{i_1ji_2} - a_{i_2ji_1} \quad \text{for any } i_1, i_2, j \in \underline{n}. \tag{2.11}$$

Taking $i_2 = j$ in (2.10) gives $a_{i_1jl} = 0$ for $i_1 \neq j$ and $l \neq j$; taking $i_1 = j$ in (2.11) gives $a_{jji_2} = 0$ for $i_2 \neq j$. It follows that

$$a_{ijl} = 0 \quad \text{for any } i, j \neq l \in \underline{n}. \tag{2.12}$$

On the other hand, it follows from (2.9) for the case $l = j_2 \neq j_1$ that

$$\delta_{j_1i_2}a_{i_1j_2j_2} - \delta_{j_2i_1}a_{i_2j_1j_2} = a_{i_1j_1i_2}, \tag{2.13}$$

which together with (2.12) gives

$$a_{i_1j_1j_1} = a_{i_1j_2j_2} \quad \text{for } i, j_1 \neq j_2 \in \underline{n}. \tag{2.14}$$

Hence by (2.12) and (2.14), the expression of $\sigma(x_i\partial_j)$ can be rewritten as

$$\sigma(x_i\partial_j) = a_i\theta_j, \ \text{ where } a_i = a_{ij_1j_1} \text{ for any } j_1 \in \underline{n}, \tag{2.15}$$

and whence (2.8) becomes

$$\theta_j(a_i) = \delta_{ij} \quad \text{for any } i, j \in \underline{n}. \tag{2.16}$$

So the Claim 1 is true.

Claim 2. *We have $\sigma(x_i^2\partial_j) = a_i^2\theta_j$ for all $i, j \in \underline{n}$.*

By (2.7) and (2.15),

$$\theta_m(a_{ijl}^{(2)}) = 2\delta_{im}\delta_{jl}a_i \quad \text{for all } i, j, l, m \in \underline{n}. \tag{2.17}$$

It follows from (2.1) and (2.15)–(2.17) that

$$\begin{aligned}
& a_{ij_1i}^{(2)}\theta_{j_2} - 2\delta_{j_2i}a_i^2\theta_{j_1} \\
= \ & \sum_{l=1}^{n}\left(a_{ij_1l}^{(2)}\theta_l(a_i)\theta_{j_2} - a_i\theta_{j_2}(a_{ij_1l}^{(2)})\theta_l\right) = \sum_{l=1}^{n}[a_{ij_1l_1}^{(2)}\theta_l, a_i\theta_{j_2}] \\
= \ & [\sigma(x_i^2\partial_{j_1}), \sigma(x_i\partial_{j_2})] = \sigma([x_i^2\partial_{j_1}, x_i\partial_{j_2}]) = \delta_{j_1i}\sigma(x_i^2\partial_{j_2}) - 2\delta_{j_2i}\sigma(x_i^2\partial_{j_1}) \\
= \ & \sum_{l=1}^{n}(\delta_{j_1i}a_{ij_2l}^{(2)} - 2\delta_{j_2i}a_{ij_1l}^{(2)})\theta_l.
\end{aligned}$$

Then by Lemma 2.1,

$$\delta_{j_1i}a_{ij_2l}^{(2)} - 2\delta_{j_2i}a_{ij_1l}^{(2)} = \delta_{j_2l}a_{ij_1i}^{(2)} - 2\delta_{j_1l}\delta_{j_2,i}a_i^2 \quad \text{for all } i, j_1, j_2, l \in \underline{n}. \tag{2.18}$$

Thus for $l \neq j_2$ and $l \neq j_1$, we have $\delta_{j_1 i} a^{(2)}_{ij_2 l} - 2\delta_{j_2,i} a^{(2)}_{ij_1 l} = 0$, which implies $a^{(2)}_{j_1 j_2 l} = 2\delta_{j_2 j_1} a^{(2)}_{j_1 j_1 l}$. Hence,

$$a^{(2)}_{j_1 j_2 l} = 0 \quad \text{for } l \neq j_1 \text{ and } l \neq j_2. \tag{2.19}$$

In case $j_1 = j_2 = j$, by (2.18) we have $-\delta_{ji} a^{(2)}_{ijl} = \delta_{jl}(a^{(2)}_{iji} - 2\delta_{ji} a_i^2)$, which gives rise to

$$a^{(2)}_{iji} = 0 \quad \text{for } i \neq j, \tag{2.20}$$

and

$$a^{(2)}_{iii} = a_i^2 \quad \text{for any } i. \tag{2.21}$$

It follows from taking $l = j_1 \neq j_2$ in (2.18) that

$$\delta_{j_1 i} a^{(2)}_{ij_2 j_1} - 2\delta_{j_2 i} a^{(2)}_{ij_1 j_1} = -2\delta_{j_2 i} a_i^2,$$

from which by setting $i = j_2$ we obtain

$$a^{(2)}_{j_2 j_1 j_1} = a_{j_2}^2 \quad \text{for } j_1 \neq j_2. \tag{2.22}$$

Now let us collect some useful datum to deduce the relation promised in Claim 2. By (2.19) and (2.20) one can see that

$$a^{(2)}_{ijl} = 0 \quad \text{for all } i, j \neq l \in \underline{n}. \tag{2.23}$$

It immediately follows from (2.21) and (2.22) that

$$a^{(2)}_{ijj} = a_i^2 \quad \text{for all } i, j \in \underline{n}. \tag{2.24}$$

Combining the above two equations gives $a^{(2)}_{ijl} = \delta_{jl} a_i^2$, and therefore

$$\sigma(x_i^2 \partial_j) = a_i^2 \theta_j \quad \text{for all } i, j \in \underline{n}. \tag{2.25}$$

This completes the proofs of Claim 2 and the lemma. $\square$

Lemma 2.3. *Let a_i be as in Lemma 2.2. We have in fact $a_i \in A_n$ for all $i \in \underline{n}$, and*

$$\sigma(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \partial_j) = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} \theta_j \quad \text{for any } j \in \underline{n}, k_i \in \mathbb{Z}^{\geq 0}.$$

Proof. First we assert $\sigma(x_i^k \partial_j) = a_i^k \theta_j$ for any $i, j \in \underline{n}$ and $k \in \mathbb{Z}^{\geq 0}$. We proceed by induction on k. By Lemma 2.2, this is true for $k \leq 2$. In particular, by (2.1), (2.7) and (2.16) we have

$$\sigma(x_l x_i \partial_i) = \frac{1}{2}\sigma([x_l \partial_i, x_i^2 \partial_i]) = \frac{1}{2}[a_l \theta_i, a_i^2 \theta_i] = a_l a_i \theta_i \quad \text{for any } l \neq i \in \underline{n}. \tag{2.26}$$

Suppose that this assertion holds for the case k. Let us see the case $k+1$. By inductive assumption, we have

$$\sigma(x_i^{k+1}\partial_j) = \frac{1}{2}\sigma([x_i^k\partial_i, x_i^2\partial_j]) = \frac{1}{2}[a_i^k\theta_i, a_i^2\theta_j] = \frac{1}{2}a_i^k\theta_i(a_i^2)\theta_j = a_i^{k+1}\theta_j \quad \text{for } i \neq j,$$

and in case $i = j$, we can always choose $l \neq i$ (since we assume $n \geq 2$) such that

$$\begin{aligned}\sigma(x_i^{k+1}\partial_i) &= \sigma([x_i^k\partial_l, x_l x_i\partial_i] + k[x_i^k\partial_i, x_i x_l\partial_l]) \\ &= [a_i^k\theta_l, a_l a_i\theta_i] + k[a_i^k\theta_i, a_i a_l\theta_l] = a_i^{k+1}\theta_i\end{aligned}$$

by (2.26). So in either case we have proved $\sigma(x_i^{k+1}\partial_j) = a_i^{k+1}\theta_j$ for any $i, j \in \underline{n}$, i.e., the assertion also holds for the case $k+1$.

Now we are going to show that $a_i \in A_n$ for all $i \in \underline{n}$. Since $\theta_i \in W_n$, we can assume that $\theta_i = \sum_{j\in\underline{n}} b_{ji}\partial_j$ for some $b_{ji} \in A_n$. Write $a_i = \frac{p_i}{q_i}$ for some coprime polynomials $p_i, q_i \in A_n$. Then noting from $\sigma(x_i^k\partial_i) = a_i^k\theta_i = \sum_{j\in\underline{n}} \frac{p_i^k}{q_i^k} b_{ji}\partial_i \in W_n$, we obtain that $q_i^k | b_{ji}$ for any $k \in \mathbb{Z}^{\geq 0}$. The only possibility for this is that $q_i \in \mathbb{C}^\times$. This shows $a_i \in A_n$.

Next by induction on r we prove $\sigma(x_{i_1}^{k_{i_1}} x_{i_2}^{k_{i_2}} \cdots x_{i_r}^{k_{i_r}}\partial_j) = a_{i_1}^{k_{i_1}} a_{i_2}^{k_{i_2}} \cdots a_{i_r}^{k_{i_r}}\theta_j$ for any $j, i_l \in \underline{n}$ and $k_{i_l} \in \mathbb{Z}^{\geq 0}$. By the first paragraph, this statement holds for $r = 1$. Suppose this holds for $1 \leq r < n$. Without loss of generality, we show that

$$\sigma(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}\partial_j) = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}\theta_j \tag{2.27}$$

provided that $\sigma(x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}}\partial_j) = a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}}\theta_j$ holds. By inductive assumption, we have

$$\begin{aligned}&\sigma(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}\partial_j) \\ =\ & -\frac{(-1)^{\delta_{nj}}}{k_{n+\delta_{nj}-1}+1}\sigma([x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}+1-\delta_{nj}}\partial_j, x_n^{k_n+\delta_{nj}}\partial_{n+\delta_{nj}-1}]) \\ =\ & -\frac{(-1)^{\delta_{nj}}}{k_{n+\delta_{nj}-1}+1}[a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}+1-\delta_{nj}}\theta_j, a_n^{k_n+\delta_{nj}}\theta_{n+\delta_{nj}-1}] \\ =\ & a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}\theta_j.\end{aligned}$$

That is, the formula (2.27) holds. This completes the proof. □

3 Proof of Theorem 1.1

Recall that an n-tuple $(f_1, f_2, \cdots, f_n)$ of elements in A_n is called a *Jacobi tuple* if the Jacobi determination $J(f_1, f_2, \cdots, f_n) = \mathrm{Det}(\partial_j f_i)_{1\leq i,j\leq n} \in \mathbb{C}^\times$.

Before beginning to prove Theorem 1.1, we also need to present the following result.

Proposition 3.1. *Any nonzero endomorphism of W_n is uniquely determined by a Jacobi tuple.*

Proof. Let $0 \neq \sigma \in \operatorname{End} W_n$. Then by Lemmas 2.2 and 2.3, there exist $f_i \in A$ and $\theta_i \in W_n$ such that

$$\sigma(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \partial_j) = f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n} \theta_j \quad \text{for all } j \in \underline{n}, k_i \in \mathbb{Z}^{\geq 0}$$

and

$$\theta_j(f_i) = \delta_{ij} \text{ for any } i, j \in \underline{n}. \tag{3.1}$$

For $j \in \underline{n}$, assume that $\theta_j = \sum_{k \in \underline{n}} a_{jk} \partial_k$ for some $a_{jk} \in A_n$. Then (3.1) is equivalent to $\sum_{k \in \underline{n}} a_{jk} \partial_k(f_i) = \delta_{ij}$, or in terms of matrix,

$$(a_{jk})_{j,k \in \underline{n}} \, (\partial_k f_i)_{k,i \in \underline{n}} = I_n \text{ (the } n \times n \text{ identity matrix).}$$

In particular, $J(f_1, f_2, \cdots, f_n) = \operatorname{Det}(\partial_k f_i)_{k,i \in \underline{n}} \in \mathbb{C}^\times$ and as the inverse matrix of $(\partial_k f_i)_{k,i \in \underline{n}}$, $(a_{jk})_{j,k \in \underline{n}}$ is uniquely determined by $(f_1, f_2, \cdots, f_n)$. This shows that σ is uniquely determined by the Jacobi tuple $(f_1, f_2, \cdots, f_n)$. $\square$

Proof of Theorem 1.1 Note that the injectivity and the surjection of ξ follow respectively from the definition (1.3) of ξ and Proposition 3.1, proving (2).

To prove (3), we only need to show that $\operatorname{Im} \zeta \subseteq \operatorname{Aut} W_n$. Since if this is true, then it is easy to see that ζ is a bijective map from $\operatorname{Aut} A_n$ onto $\operatorname{Aut} W_n$. Note that any nonzero element of $\operatorname{End} W_n$ is injective (cf. Section 1). So it is enough to show that for any given $\tau \in \operatorname{Aut} A_n$, the image σ_{f_τ} of τ under the map ζ is surjective in $\operatorname{End} W_n$. Assume $J(\tau(x_1), \tau(x_2), \cdots, \tau(x_n)) = c \in \mathbb{C}^\times$. Let M^* be the adjoint matrix of $M = \left(\frac{\partial \tau(x_i)}{\partial x_j}\right)_{1 \leq i,j \leq n}$. Define $\theta_j \in W_n$ for $j \in \underline{n}$ in the following way

$$(\theta_1, \theta_2, \cdots, \theta_n)^T = \frac{1}{c} M^* (\partial_1, \partial_2, \cdots, \partial_n)^T.$$

Here the symbol T stands for the transpose. Then by the definition of σ_{f_τ} (cf. (1.2)), we have

$$\sigma_{f_\tau}(h \partial_j) = \tau(h) \theta_j \quad \text{for all } j \in \underline{n} \text{ and } h \in A_n. \tag{3.2}$$

Since M is non-degenerate, so is M^* and thereby each ∂_i is an A_n-linear combination of θ_j's, say,

$$\partial_i = \sum_{j \in \underline{n}} b_{ji} \theta_j \tag{3.3}$$

for some $b_{ji} \in A_n$. Thus, $\partial_i = \sigma_{f_\tau}\Big(\sum_{j \in \underline{n}} \tau^{-1}(b_{ji}) \partial_j\Big) \in \operatorname{Im} \sigma_{f_\tau} \subseteq W_n$ for any $i \in \underline{n}$.

On the other hand, by (3.2) and (3.3) one can see that

$$\sigma_{f_\tau}\Big(\tau^{-1}(x_i^2) \sum_{k \in \underline{n}} \tau^{-1}(b_{kj}) \partial_k\Big) = x_i^2 \sum_{k \in \underline{n}} b_{kj} \theta_k = x_i^2 \partial_j.$$

In particular, $x_i^2\partial_j \in \operatorname{Im}\sigma_{f_\tau} \subseteq W_n$ for any $i, j \in \underline{n}$. Thus we have obtained

$$\{x_i^2\partial_j, \partial_j \mid i, j \in \underline{n}\} \subseteq \operatorname{Im}\sigma_{f_\tau} \subseteq W_n.$$

This forces $W_n = \operatorname{Im}\sigma_{f_\tau}$ since $\{x_i^2\partial_j, \partial_j \mid i, j \in \underline{n}\}$ is a generating set of the Lie algebra W_n (recall that we assume $n \geq 2$, cf. Lemma 2.3). This shows the surjection of σ_{f_τ}.

As we have mentioned, the Jacobi conjecture following from the Witt algebra's conjecture was proved in [22, Theorem 4.1], so for (1) it remains to show that the Jacobi conjecture implies the Witt algebra's conjecture. Let $\phi \in \operatorname{End} W_n \backslash \{0\}$. We have to show $\phi \in \operatorname{Aut} W_n$. By (2), ϕ corresponds to a Jacobi tuple, say, $\xi^{-1}(\phi) = f_\phi = (f_{\phi 1}, f_{\phi 2}, \cdots, f_{\phi n})$. This Jacobi tuple induces an endomorphism τ of A_n defined by

$$\tau(x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}) = f_{\phi 1}^{k_1}f_{\phi 2}^{k_2}\cdots f_{\phi n}^{k_n} \quad \text{for any } k_l \in \mathbb{Z}^{\geq 0}.$$

Now it follows from the equivalent statement of the Jacobi conjecture as remarked in Section 1 that $\tau \in \operatorname{Aut} A_n$. So by (3), $\sigma_{f_\tau} = \zeta(\tau) \in \operatorname{Aut} W_n$, where $f_\tau = f_\phi$. Then it follows from (2) that $\phi = \sigma_{f_\phi} = \sigma_{f_\tau} \in \operatorname{Aut} W_n$, as desired.

Note that (4) follows immediately from (3) and the fact that $\operatorname{Aut} A_2$ is generated by $s = s_1$, τ_a (for $a \in \mathbb{C}^\times$) and ψ_p (for $p \in \mathbb{Z}^{\geq 0}$) (cf. Section 1 and [6]). This completes the proof Theorem 1.1.

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